Optimal investment strategies and hedging of derivatives in the presence of transaction costs

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ABSTRACT

Investment strategies in multiplicative Markovian market models with transaction costs are defined using growth optimal criteria. The optimal strategy is shown to consist in holding the amount of capital invested in stocks within an interval around an ideal optimal investment. The size of the holding interval is determined by the intensity of the transaction costs and the time horizon. The inclusion of financial derivatives in the models is also considered. All the results presented in this contributions were previously derived in collaboration with E. Aurell in papers[3–5]

Keywords: Statistical finance; Markov process; transaction costs; stochastic optimization; Black and Scholes.

1. INTRODUCTION

An idealised model of investment is a sequence of gambles where the speculator chooses at each time step her position. The game is multiplicative if the pay-off is proportional to the capital, and it is Markov if the new capital and new position depend parametrically only on the previous state. The relevant issue consists of determining which strategy the speculator should pursue. A reasonable choice is to assume that the investor wishes to maximise the growth of her capital. This latter quantity lends itself to defining a utility function which permits to specify the growth-optimal investment as a function of the parameters of the model.

Growth optimal criteria for multiplicative Markov process were first investigated by Kelly in the context of information theory.[17] For recent reviews the reader is referred to ref's.[2, 16, 18] "Universal portfolios" procedures[10, 11] can also be considered as examples of application growth-optimal criteria.

In the present contribution the issue of how optimal growth strategies are affected by transaction costs is addressed. Following ref's,[3, 4] a market model is costructed as the continuous time limit of a discrete multiplicative Markov game describing the dynamics of a stock and bond portfolio. Trading costs are modelled either by a linear (i.e. proportional to the absolute value of the capital moved by the investor to balance her portfolio) or by quadratic function of the fraction of capital invested in re-hedging the portfolio. While linear transaction costs seems to corresponds to more common financial situations, quadratic transaction costs lead to a mathematical model essentially solvable analytically, a fact which is interesting in itself. From the financial side, quadratic frictions naturally arise as a particular case of the market impact phenomenology of Farmer.[12] In such framework, if one assumes that market depth grows proportionally to the total wealth of a typical investor in the market one recovers the quadratic model presented here. The analysis of quadratic transaction costs can therefore, for example, be relevant to fairly large operators in a market, the actions of which move market prices, to some extent.

From the mathematical point of view, the continuos time models are expressed in the form of a system of coupled stochastic differential equations, governing the dynamics of the overall investor capital and of the fraction of capital invested in stocks. These quantities depends upon control functionals describing the investment

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strategy. Controls are determined by solving an Hamilton-Jacobi-Bellman equation[14, 19] giving the optimal growth rate of the investor capital.

The analysis carried out in papers[3, 4] evinces that optimal trading strategies in the presence of linear and quadratic trading costs are qualitatively identical. The results of[3, 4] can be summarised by saying that on an infinite time horizon the investment optimal strategy consists of allowing the amount of capital invested in stocks to fluctuate freely within an interval around the optimal portfolio as determined *in the absence* of trading costs. The size of the holding interval depends non-analytically on an adimensional parameter measuring the intensity of the transaction costs.

For financially reasonable values of the parameters in the model, convergence to the full dynamical solution in the infinite time horizon limit may be surprisingly slow, of the order of years of trading. Hence it is very relevant the question of the time evolution of the growth optimal strategy over finite investment horizons.

Inquiring the finite investment horizon dynamics has relevance also in the perspective of deriving an option pricing procedure from growth optimal criteria. In ref[5] it was shown that for the market models described in ref's[3, 4] the well-posedness of the optimisation problem in the presence of derivatives requires to impose a solvability conditions. The Black and Scholes equation[6, 7] was then shown to provide a natural way to impose such solvability conditions by ruling out the possibility of self-financing portfolios.

The paper is organised as follows. In section 2 the continuous time market model is derived from an elementary discrete time dynamics. In section 3 the optimisation problem is stated in the framework of the Hamilton-Jacobi-Bellman equation. In section 4 using elementary dimensional consideration it is shown that the Hamilton-Jacobi-Bellman equation is solvable in the small transaction costs limit by means of a multi-scale perturbation theory.[9, 13, 15]

In section 5 financial derivatives are included in the market model. The emergence[5] of the Black and Scholes equation as a solvability condition for the optimisation problem is discussed. The qualitative properties of the optimum investment and hedging strategy are also shortly illustrated.

The last section is devoted to conclusions.

2. DERIVATION OF THE MODEL

Consider an investor endowed at time t with a capital W_t a fraction ρ_t whereof is invested in stocks

$$W_t^{(Stocks)} = \rho_t \, W_t \tag{1}$$

The variation in one time step of the wealth in stocks occurs in consequence of

- the market fall-out u_t
- the action $\Delta\chi_t$ of the speculator who re-hedges her position in the market.

The fraction in stocks at time $t+1$ becomes

$$W_{t+1}^{(Stocks)} = [u_t \rho_t + \Delta\chi_t] \; W_t \tag{2}$$

The total wealth at time $t+1$ is affected by the stock investment profits or losses and by the trading costs entailed by any re-hedging:

$$W_{t+1} = [1 + \rho_t(u_t - 1) - \Delta F_\gamma(\Delta\chi_t)] \; W_t \tag{3}$$

Most generally trading costs are described by a semi-positive definite function ΔF_γ vanishing only if the investor remains idle, i.e. when $\Delta\chi$ is zero.
From (2), (3) the variation of the invested capital fraction ρ_t over a time unit is

$$\Delta\rho_t = \frac{\rho_t + (u_t - 1)\rho_t + \Delta\chi_t}{1 + \rho_t(u_t - 1) - \Delta F_\gamma(\Delta\chi_t)} - \rho_t \tag{4}$$

The continuum limit is attained by replacing

$$\begin{aligned} u_t - 1 &\rightarrow \mu\, dt + \sigma\, dB_t \\ \Delta\chi_t &\rightarrow f\, dt \\ \Delta F &\rightarrow \gamma\mathcal{F}(f)dt \end{aligned} \tag{5}$$

The stochastic control f represents the action taken by the investor at time t to re-hedge her position in the market. The differential du_t gives the relative stock price

$$du_t := \frac{dS_t}{S_t} = \mu\, dt + \sigma\, dB_t \tag{6}$$

The stochastic differential equation is defined according to the Ito convention. It has the solution

$$S_t = S_o\, e^{\left(\mu - \frac{\sigma^2}{2}\right)\, t + \sigma\, B_t} \tag{7}$$

A value of $\frac{\mu}{\sigma^2}$ outside the interval $[0, 1]$ thus corresponds to strong inflation or deflation rates. If borrowing and short-selling is not allowed, the optimal strategy would then simply be to keep all money in stock or all money in bonds. If borrowing and short-selling is allowed, the problem becomes again similar to the one studied here, but the relevant intervals would then either be $[1, \infty]$ or $[-\infty, 0]$.

After a little algebra one finds

$$dW_t = [\mu\rho_t - \gamma f^2(\rho_t, t)]\, W_t dt + \sigma\rho_t W_t\, dB_t \tag{8}$$

$$d\rho_t = \left[f(\rho_t, t) + a(\rho_t) + \gamma\rho_t f^2(\rho_t, t)\right]\, dt + b(\rho_t)\, dB_t \tag{9}$$

with

$$\begin{aligned} a(\rho_t) &= \rho_t(1 - \rho_t)(\mu - \sigma^2\rho_t) \\ b(\rho_t) &= \sigma\rho_t(1 - \rho_t) \end{aligned} \tag{10}$$

The stochastic control is determined as a function of ρ_t and t, by maximising the expectation value of the wealth growth:

$$\lambda(x, t; T) = E_{\rho_t = x} \ln \frac{W_T}{W_t} = E_{\rho_t = x} \int_t^T ds\, [V(\rho_s) - \gamma\mathcal{F}] \tag{11}$$

with

$$V(\rho_t) = \mu\rho_t - \frac{\sigma^2}{2}\rho_t^2 \tag{12}$$

The expectation $E_{\rho_t = x}$ is conditional on the fraction in stock process ρ_t having value x at initial time t. The time difference $T - t$ is the time horizon of the speculator: the time period wherein she wants to optimise her position in the market. The optimisation is performed with respect to two conflicting effects. On one hand, the market fall-outs raise or lower the relative amount of invested wealth, motivating the investor to re-balance the portfolio. On the other, the re-hedging carries trading costs. Two cases will be considered

- *"linear"* trading costs $\mathcal{F} = |f|$
- *"quadratic"* trading costs $\mathcal{F} = f^2$

In both cases γ is some given positive valued constant.

3. HAMILTON-BELLMAN-JACOBI FORMALISM

For any stochastic control f such that the system (8), (9) is well defined, the expectation value of the wealth growth must obey the dynamic programming equation

$$\begin{aligned} &\partial_t \lambda + [f + a + \gamma x \mathcal{F}] \partial_x \lambda + \frac{b^2}{2} \partial_x^2 \lambda + V - \gamma \mathcal{F} = 0 \\ &\lambda(x, T; T) = 0 \end{aligned} \tag{13}$$

If borrowing and short-selling is not allowed, the probability measure of ρ is conserved in the interval $[0, 1]$. This means for λ (see also[4]):

$$\partial_x \lambda(x, t; T)|_{x=0} = \partial_x \lambda(x, t; T)|_{x=1} = 0 \tag{14}$$

It is useful to identify the canonical dimensions of the quantities involved in the problem:

$$[\lambda] = [x] = 0\,, \quad [\sigma^2] = [\mu] = [f] = [1/t]\,, \quad [\gamma \mathcal{F}] = [1/t] \tag{15}$$

The optimal growth condition corresponds to an extremal point in the functional dependence of λ upon the stochastic control f.[14, 19] The detailed form of the extremum condition depends on the mathematical modelling of transaction costs. For "linear" trading costs it was found[3]

$$\begin{aligned} &\partial_t \lambda + a \partial_x \lambda + \tfrac{b^2}{2} \partial_x^2 \lambda + V = 0 \\ &(\partial_x \lambda)(x_{Max}, t; T) = -\tfrac{\gamma}{1 - \gamma x_{Max}} \\ &(\partial_x \lambda)(x_{min}, t; T) = \tfrac{\gamma}{1 + \gamma x_{min}} \end{aligned} \tag{16}$$

i.e. the stochastic process ρ_t associated to optimal growth is confined within an interval $[x_{min}\,, x_{Max}]$ varying in time inside $[0, 1]$ with a probability measure which is not absolutely continuous at the boundary.[14]

Quadratic friction provides for a smoother extremum condition

$$\begin{aligned} &\partial_t \lambda + a\, \partial_x \lambda + \tfrac{(\partial_x \lambda)^2}{4\, \gamma\, (1 - x\, \partial_x \lambda)} + \tfrac{b^2}{2} \partial_x^2 \lambda + \mu x - \tfrac{\sigma^2 x^2}{2} = 0 \\ &\lambda(x, T; T) = 0 \\ &\partial_x \lambda(x, t; T)|_{x=0} = \partial_x \lambda(x, t; T)|_{x=1} = 0 \end{aligned} \tag{17}$$

In view of Odeledec's theorem[20] the solution of both (16) and (17) can be sought in the form

$$\lambda(x, t) = (T - t)\, \ell + \bar{\lambda}(x, t) \tag{18}$$

with ℓ specifying the asymptotic optimal growth rate.

4. NORMAL FORMS OF THE HAMILTON-BELLMAN-JACOBI EQUATIONS

In the absence of transaction costs the speculator is free to take un-restrained actions to always keep the fraction allocated to stocks constant

$$\rho^{\text{opt}} = \frac{\mu}{\sigma^2} \tag{19}$$

The optimal growth rate of the investor capital is[1, 19]:

$$\ell|_{\gamma=0} = \frac{\mu^2}{2\, \sigma^2} \tag{20}$$

corresponding to the solution

$$\lambda(x, t) = (T - t)\, \ell|_{\gamma=0} \tag{21}$$

of the Hamilton-Bellman-Jacobi equation.

Elementary scaling considerations permit to shed some light on the behaviour of the solutions of (16) and (17) in the presence of a small but finite trading costs. It is convenient to translate the origin of the x coordinate to the value of the ideal optimum investment

$$x \to x + \frac{\mu}{\sigma^2} \tag{22}$$

The change of variables permits to rewrite drift and diffusion terms in the Hamilton-Bellman-Jacobi equations as

$$\begin{aligned} a &= \delta a\,, & \delta a &= \textstyle\sum_{i=1}^{2} a_i x^i \\ b &= D + \delta b\,, & \delta b &= \textstyle\sum_{i=1}^{2} b_i x^i \end{aligned} \tag{23}$$

with

$$D := \frac{\mu}{\sigma}\left(1 - \frac{\mu}{\sigma^2}\right)$$

The explicit form of the polynomial coefficients is not relevant for the following discussion and it is straightforwardly derived by inserting (22) into (10). Furthermore in the new coordinates specified by (22)

$$\delta V = V - \ell|_{\gamma=0} = \frac{\sigma^2 x^2}{2} \tag{24}$$

4.1. Quadratic transaction costs

Let us first consider the case of quadratic friction. The adimensional parameter measuring the intensity of trading costs is

$$\epsilon = \sigma^2 \gamma \tag{25}$$

The solution of (17) can always be couched into the form

$$\lambda(x,t) = (T-t)\ell|_{\gamma=0} + \varphi(x,t;\epsilon) \tag{26}$$

where φ now satisfies

$$\partial_t \varphi + \delta a \partial_x \varphi + \frac{\sigma^2 (\partial_x \varphi)^2}{4\,\epsilon} + \frac{\sigma^2 (\rho^{\rm opt} + x)(\partial_x \varphi)^3}{4\,\epsilon\,[1 - (\rho^{\rm opt} + x)\partial_x \varphi]} + \frac{(D+\delta b)^2}{2}\partial_x^2 \varphi + \delta V = 0 \tag{27}$$

This latter equation can be rescaled according to

$$\varphi \Rightarrow \epsilon^{\omega_\varphi}\,\varphi\,, \qquad x \Rightarrow \epsilon^{\omega_x}\,x\,, \qquad t \Rightarrow \epsilon^{\omega_t}\,t \tag{28}$$

Neglecting δa and δb as higher order, the requirement

$$\omega_\varphi \geq \omega_x > 0 \tag{29}$$

enforcing the existence of the limit ϵ tending to zero and its convergence to (20) of λ gives

$$\omega_\varphi = 1\,, \qquad \omega_x = \frac{1}{4}\,, \qquad \omega_t = \frac{1}{2} \tag{30}$$

The conclusion is that the normal form[9, 13] of the non-linearity in (17) is

$$\begin{aligned} &\partial_t \lambda + \frac{(\partial_x \lambda)^2}{4\,\gamma} + \frac{D^2}{2}\partial_x^2 \lambda + \frac{\mu^2}{2\,\sigma^2} - \frac{\sigma^2 x^2}{2} = 0 \\ &\lambda(y,T;T) = 0 \\ &\lambda(x,t;T)\,|_{x=-\infty} = \lambda(x,t;T)\,|_{x=\infty} = 0 \end{aligned} \tag{31}$$

Note that for small ϵ the properties of the "bulk" of λ can be approximated by imposing probability conservation on the entire real axis, as the solution is expected to decay rapidly as $|x|$ increases. Equation (31) is exactly solvable.[4] It yields an asymptotic optimal-growth rate

$$\ell = \frac{\mu^2}{2\,\sigma^2} - \frac{D^2\,\sigma^2\,\tau}{2}$$

with

$$\tau = \sqrt{\frac{2\,\gamma}{\sigma^2}} \equiv \frac{2}{\sigma^2}\sqrt{\frac{\epsilon}{2}} \tag{32}$$

The investment strategy as a function of the horizon is most conveniently described in terms of a control potential U

$$f(x, T-t) = -\partial_x U(x, T-t) \tag{33}$$

The integration constant can be fixed by setting

$$U(x, T-t) = -\frac{\lambda(x, T-t)}{2\,\gamma} + \frac{\lambda(0, T-t)}{2\,\gamma} \tag{34}$$

The behaviour in time of the potential is illustrated in figure 1. The potential gets steeper for long investment horizons tending asymptotically to a parabolic shape

$$U_{\text{asympt.}}(x) = \frac{x^2}{2\tau}\,, \qquad (T-t) \gg \tau \tag{35}$$

In this limit the speculator aims to always hold the invested fraction of capital in a finite interval around the optimal investment fraction ρ^{opt} of (19). In the asymptotic regime the fraction of capital invested in stocks tends to an Ornstein-Uhlenbeck process[8] the invariant measure whereof having variance $D^2\tau/2$. This latter quantity provides the typical size

$$L^\star = \sqrt{D^2\tau} \propto \epsilon^{\frac{1}{4}} \tag{36}$$

of the holding interval in which the speculator allows asymptotically to fluctuate freely. The convergence to the asymptotic regime is exponential with decay rate $\tau/2$. Note that functional dependence of $L^\star$ and τ upon ϵ can be inferred without explicitly solving (31) from the scaling dimensions (30).

4.2. Linear transaction costs

Analogous scaling considerations can be repeated for (17) with the proviso that the adimensional measure ϵ of the intensity of trading costs coincides now with γ for dimensional reasons. One finds

$$\omega_\varphi = \frac{4}{3}\,, \qquad \omega_x = \frac{1}{3}\,, \qquad \omega_t = \frac{2}{3} \tag{37}$$

Correspondingly, the normal form of the equation is

$$\begin{aligned} &\partial_t\lambda + \frac{D^2}{2}\partial_x^2\lambda + \frac{\mu^2}{2\,\sigma^2} - \frac{\sigma^2 x^2}{2} = 0 \\ &\lambda(y, T; T) = 0 \\ &(\partial_x\varphi)(x_{Max}, t; T) = -(\partial_x\varphi)(x_{min}, t; T) = -1 \end{aligned} \tag{38}$$

which was studied in ref..[3] The concept of holding interval becomes sharp in the case of linear trading costs. The probability measure of the fraction of capital invested in stocks is non-vanishing only in the interval comprised between x_{min} and x_{Max}. The asymptotical size of the interval is for small ϵ

$$L^\star = (12\,D^2\,\epsilon)^{\frac{1}{3}} \tag{39}$$

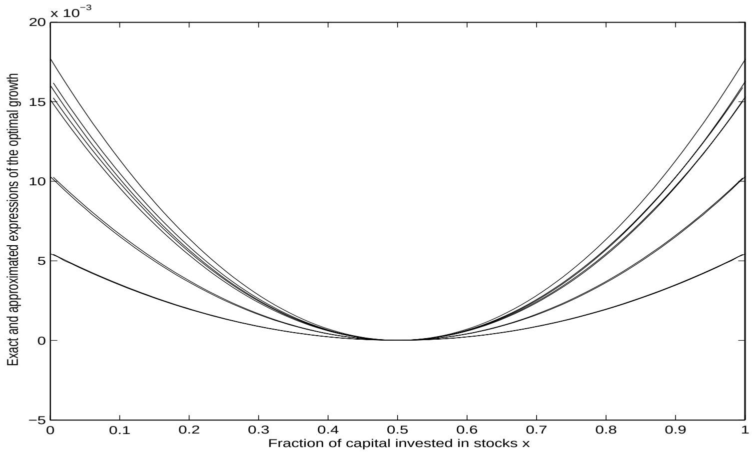

Figure 1. The control potential $2\gamma V$ as defined in (34) for $\mu = \sigma^2/2$, $\sigma = 10^{-2}$ and $\gamma = 10^2$ is plotted for time horizons of $T - t = 500, 1000, 2000, 2500$ days using both the exact solution of the Hamilton-Jacobi-Bellman equation (17) and of its normal form (31). The characteristic decay time to the asymptotic regime is $\tau/2 \sim 1000$ days. The innermost parabola is obtained from the asymptotic expression (35).

while the asymptotic regime is attained exponentially with a decay rate

$$\tau \propto \epsilon^{\frac{2}{3}} \tag{40}$$

Again the scaling (39), (40) with ϵ (i.e. γ) can be read a priori from the scaling dimensions (37).

The qualitative behaviour of the control strategy versus the investment horizon is similar to the smooth case of quadratic trading costs. The holding interval shrinks as the the time horizon increases. The time dependence is non-analytic rendering the construction of explicit solution of the non-stationary regime technically more demanding. Details on the finite horizon behaviour of the linear transaction costs model can be found in ref..[5]

5. FINANCIAL DERIVATIVES

The market model discussed in the previous sections can be generalised to include financial derivatives. This was done in ref..[5] In order to simplify the discussion it is convenient to restrict the attention to the case of quadratic transaction costs.

In the presence of a financial derivative the market model is described by the set of four stochastic differential equations

$$\begin{aligned}
dW_t &= W_t \left[\left(\mu\rho_t + \mu^{(d)}\eta_t \right) dt + \left(\sigma\rho_t + \sigma^{(d)}\eta_t \right) dB_t - \left(\gamma\, f^2 + \gamma^{(d)}\, {f^{(d)}}^2 \right) dt \right] &(41)\\
d\rho_t &= \left[f + a^{(s)} + \rho_t \left(\gamma f^2 + \gamma^{(d)}\, {f^{(d)}}^2 \right) \right] dt + b^{(s)}\, dB_t &(42)\\
d\eta_t &= \left[f_d + a^{(d)} + \eta_t \left(\gamma f^2 + \gamma^{(d)}\, {f^{(d)}}^2 \right) \right] dt + b^{(d)}\, dB_t &(43)\\
dS_t &= \mu S_t\, dt + \sigma\, S_t dB_t &(44)
\end{aligned}$$

where

$$a^{(s)} = \mu\, \rho_t - \rho_t \left[\mu\rho_t + \mu^{(d)}\eta_t - \left(\sigma\rho_t + \sigma^{(d)}\eta_t - \sigma \right) \left(\sigma\rho_t + \sigma^{(d)}\eta_t \right) \right] \tag{45}$$

$$a^{(d)} = \mu^{(d)}\, \eta_t - \eta_t \left[\mu\rho_t + \mu^{(d)}\eta_t - \left(\sigma\rho_t + \sigma^{(d)}\eta_t - \sigma^{(d)} \right) \left(\sigma\rho_t + \sigma^{(d)}\eta_t \right) \right] \tag{46}$$

and

$$b^{(s)} = \sigma\,\rho_t - \rho_t\,\left(\sigma\rho_t + \sigma^{(d)}\eta_t\right) \tag{47}$$

$$b^{(d)} = \sigma^{(d)}\,\eta_t - \eta_t\,\left(\sigma\rho_t + \sigma^{(d)}\eta_t\right) \tag{48}$$

Some remarks are in order. The equation for the stock price dynamics (44) is coupled to the system through the drift and diffusion fields $\mu^{(d)}$ and $\sigma^{(d)}$. These quantities are specified by the derivative price. In ref.[5] the derivative price C was taken to be a pure function of the underlying price alone:

$$\frac{dC}{C} = \frac{1}{C}\left[\partial_t C + \mu S\partial_S C + \frac{\sigma^2\,S^2}{2}\partial_S^2 C\right] dt + \frac{\sigma S\partial_S C}{C}\,dB_t := \mu^{(d)}dt + \sigma^{(d)}\,dB_t \tag{49}$$

Hence $\mu^{(d)}$ and $\sigma^{(d)}$ are known functions of S whenever C is given. Thus, because of derivative trading described by η_t, the fraction of capital invested in the financial derivatives, the equations depend explicitly upon the underlying price at variance with the stock and bond model where the dynamical equations depended solely on the relative change in price of the stock. All these quantity are affected by the same market fall-out mocked-up by increments dB_t over realisations of the *same* Brownian motion B_t. Finally, trading of derivatives implies a transaction costs proportional to $\gamma^{(d)}$.

The investment strategy is described in (41), (42), (43) by a the pair of stochastic controls $f\,,f^{(d)}$. Optimal growth criteria suggest to determine these parameters by requiring the exponential growth

$$\lambda(x,t;T) = E_{\rho_t=x}\ln\frac{W_T}{W_t} = E_{\rho_t=x}\int_t^T ds\,\left[V^{(d)}(\rho_s\,,\eta_s) - \gamma f^2 - \gamma^{(d)}{f^{(d)}}^2\right] \tag{50}$$

with

$$V^{(d)}(\rho\,,\eta) = \mu\,\rho + \mu^{(d)}\,\eta - \frac{(\sigma\,\rho + \sigma^{(d)}\,\eta)^2}{2} \tag{51}$$

to attain its supremum value. From such surmise, in the absence of transaction costs (51) should determine the optimal investment strategy. However, the presence of a derivative introduces a significant novelty in the problem. The potential $V^{(d)}$ is a degenerate quadratic functional of ρ and η: the Hessian of (51) has a zero eigenvalue. From the financial point of view the phenomenon, signals the possibility of *arbitrage.* For example, choices of $\mu^{(d)}$ and $\sigma^{(d)}$ such that

$$\mu - \frac{\sigma}{\sigma^{(d)}}\mu^{(d)}\ >\ 0 \tag{52}$$

permit to construct with probability one a self-financing portfolio by *going short* on derivatives while allocating in stocks a fraction of capital such that ρ and η are assigned on the marginal subspace of the Hessian of $V^{(d)}$. Such portfolio returns values of $V^{(d)}$ linearly growing ρ:

$$V_\star = \rho\left(\mu - \frac{\sigma}{\sigma^{(d)}}\mu^{(d)}\right)\ >\ 0 \tag{53}$$

In consequence the optimisation problem is ill-defined for a generic choice of $\mu^{(d)}$ and $\sigma^{(d)}$.

Growth optimum criteria can be invoked if a solvability condition in form of a *no-arbitrage* requirement is imposed on the market model. The discussion of the no-arbitrage condition is simplified by the introduction of a portfolio variable defined as

$$\zeta = \rho + \frac{\sigma^{(d)}}{\sigma}\,\eta \tag{54}$$

Adopting the new variable

$$V_d(\rho\,,\eta) = \mu\zeta - \frac{\sigma^2\zeta^2}{2} + \left(\frac{\sigma^{(d)}}{\sigma}\mu - \mu^{(d)}\right)\,\eta \tag{55}$$

becomes a convex functional of ζ provided

$$\mu - \frac{\sigma}{\sigma^{(d)}}\mu^{(d)} = 0 \tag{56}$$

Comparison with (49) shows that (56) is equivalent to

$$\partial_t C + \frac{\sigma^2 S^2}{2}\partial_S^2 C = 0 \tag{57}$$

which is the celebrated Black and Scholes equation[6, 7] determining the fair price of a derivative in an arbitrage free market model. Note that in the derivation of the market model, the risk-free rate of return was assumed to be zero.

5.1. Optimal capital growth and hedging of derivatives

Imposing the no-arbitrage condition (56) permits to eliminate $\mu^{(d)}$ from the equations. Furthermore, the dynamical equation for the portfolio variable ζ

$$d\zeta = \left[f + \frac{\sigma^{(d)}}{\sigma}f^{(d)} + a^{(p)} + \zeta\left(\gamma\, f^2 + \gamma^{(d)} {f^{(d)}}^2\right)\right]dt + b^{(p)}\,dB_t \tag{58}$$

can be used instead of (42). The drift and diffusion fields depend upon the Black and Scholes derivative price

$$a^{(p)} = a + \eta\left[\sigma^{(d)}\left(\frac{\sigma^{(d)}}{\sigma} - 1\right)\left(\mu - \sigma^2\zeta\right) + H + \left(\sigma^{(d)} - \sigma\zeta\right)\, K\right] \tag{59}$$

$$a^{(d)} = \eta\left(\frac{\sigma^{(d)}}{\sigma} - \zeta\right)\left(\mu - \sigma^2\zeta\right) \tag{60}$$

$$b^{(p)} = b + \eta\left[\sigma^{(d)}\left(\frac{\sigma^{(d)}}{\sigma} - 1\right) + K\right] \tag{61}$$

$$b^{(d)} = \eta\,(\sigma^{(d)} - \sigma\,\zeta) \tag{62}$$

with a, b given by (10) and

$$H := \frac{1}{\sigma}\left[\frac{\partial\sigma^{(d)}}{\partial t} + \mu S_t\partial_{S_t}\sigma^{(d)} + \frac{(\sigma S_t)^2}{2}\partial_{S_t}^2\sigma^{(d)}\right], \qquad K := S_t\partial_{S_t}\sigma^{(d)} \tag{63}$$

The resulting Hamilton-Bellman equation is

$$\partial_t\lambda + a^{(p)}\partial_z\lambda + a^{(d)}\partial_y\lambda + \sigma S\partial_S\lambda + \frac{\left[\sigma^2 + \tilde{\gamma}({\sigma^{(d)}}^2 - \sigma^2)\right](\partial_z\lambda)^2}{4\,\varepsilon\,(1 - z\partial_z\lambda - y\partial_y\lambda)} + \tilde{\gamma}\frac{2\,\sigma\,\sigma^{(d)}(\partial_z\lambda)(\partial_y\lambda) + (\sigma\partial_y\lambda)^2}{4\,\varepsilon\,(1 - z\partial_z\lambda - y\partial_y\lambda)}$$
$$+ \frac{{b^{(p)}}^2}{2}\partial_z^2\lambda + \frac{{b^{(d)}}^2}{2}\partial_y^2\lambda + \frac{\sigma^2 S^2}{2}\partial_S^2\lambda + b^{(p)}b^{(d)}\partial_{z\,y}\lambda + b^{(p)}\,\sigma S\partial_{z\,S}\lambda + b^{(d)}\,\sigma\,S\partial_{y\,S}\lambda + \mu z - \frac{\sigma^2 z^2}{2} = 0 \tag{64}$$

Boundary conditions of interest should now allow to go long or short in derivatives. Consequently probability conservation may be imposed on the real axis for both the (y, z) variables.

In (64) there appear two adimensional parameters measuring the intensity of transaction costs

$$\varepsilon = \sigma^2\frac{\gamma\gamma^{(d)}}{\gamma + \gamma^{(d)}} \tag{65}$$

and

$$\tilde{\gamma} = \frac{\gamma}{\gamma + \gamma^{(d)}} \tag{66}$$

The first quantity is the analogous of the order parameter (25). The existence of a well defined limit as ε goes to zero dictates the form of the multi-scale perturbation theory providing the asymptotic solution of the optimal growth problem (64). The second order parameter (66) measures instead the relative importance of transaction costs of uderlying and derivatives trading. If transaction costs in stocks and derivatives go to zero with the same speed then $\tilde{\gamma}$ tends to a finite value in such limit.

Equation (64) can be solved by means of multi-scale perturbation theory.[5] Some results are briefly reported here. Suppose that the derivative is an European call option, a classical example in mathematical finance. Then if by $S^\star$ is denoted the strike price,

$$\sigma^{(d)} = \frac{\sigma\, S\, N(\phi_1)}{S\, N(\phi_1) - S^\star\, N(\phi_2)}$$
$$\phi_1 := \frac{\ln\left(\frac{S}{S^\star}\right) + \frac{\sigma^2\,(T-t)}{2}}{\sigma\,\sqrt{T-t}}\,, \qquad \phi_2 := \phi_1 - \sigma\,\sqrt{T-t} \tag{67}$$

and

$$N(x) = \int_{-\infty}^{x} dy\, \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\,\pi}} = \frac{1}{2}\left[1 + \mathrm{Erf}\left(\frac{x}{\sqrt{2}}\right)\right] \tag{68}$$

For $S \gg S^\star$ and finite horizon or for large investment horizons

$$\sigma^{(d)} \to \sigma \tag{69}$$

while H and K tend to zero. Then scaling analysis in ε yields the normal form

$$\partial_t\lambda + \frac{\sigma^2(\partial_z\lambda)^2}{4\,\epsilon} + \frac{D^2}{2}\partial_z^2\lambda + \mu z - \frac{\sigma^2 z^2}{2} = 0 \tag{70}$$

In consequence the asymptotic growth rate becomes

$$\ell = \frac{\mu^2}{2\,\sigma^2} - \frac{D^2}{2}\sqrt{\frac{\varepsilon}{2}} \tag{71}$$

Since for any γ according to the definitions (25) and (65) the inequality $\varepsilon < \epsilon$ holds true, composing a portfolio of stocks and derivatives brings about an increase of the asymptotic capital growth rate.

6. CONCLUSIONS

In the present paper two models of trading with transaction costs were analysed using dynamic programming techniques. The results of such analysis support an “investment confinement” picture as growth optimal strategy for multiplicative Markov market models with trading costs. According to such picture, differences in the modelling of the trading costs are reflected only in the different non-analytic powers of the adimensional parameter measuring the intensity of transaction costs on which the size of the holding interval depends.

Furthermore the role of financial derivatives was also taken into account in the context of multiplicative Markov market models with trading costs. Black and Scholes pricing of derivatives was shown to emerge as a natural solvability condition for growth optimal criteria. These latter provide through the solution of an Hamilton-Bellman-Jacobi equation the hedging strategy that the investor should pursue to maximise her profits.

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